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Abstract. We define a monodromy homomorphism for irreducible families of regular elliptic fibrations which takes values in the mapping class group of a punctured sphere. In the computation we consider only elliptic fibrations which contain no singular fibres of types other than I_1 and I_0^* .

We compare the maximal groups, which can be the monodromy groups of algebraic, resp. differentiable families of elliptic surfaces, and give an algebraic criterion for the equality of both groups which we can check to apply in case the number of I_1 fibres is at most 6.

introduction

The monodromy problems we want to discuss fit quite nicely into the following general scheme: Given an algebraic object X consider an algebraic family $g : \mathcal{X} \rightarrow T$ such that a fibre $g^{-1}(t_0)$ is isomorphic to X and such that the restriction to a connected subfamily $g| : \mathcal{X}' \rightarrow T'$ containing X is a locally trivial differentiable fibre bundle. If G is the structure group of this bundle, the geometric monodromy is the natural homomorphism $\rho : \pi_1(T', t_0) \rightarrow G$. A monodromy map with values in a group A is obtained by composition with some representation $G \rightarrow A$.

In the standard setting X is a complex manifold, e.g. a smooth complex projective curve. In this case \mathcal{X} is a flat family of compact curves containing X , the subfamily \mathcal{X}' contains only the smooth curves and is a locally trivial bundle of Riemann surfaces with structure group the mapping class group $Map(X)$. From the geometric monodromy one can obtain the algebraic monodromy by means of the natural representation $Map(X) \rightarrow Aut(H_1(X))$.

In the present paper we investigate the monodromy of regular elliptic fibrations. So X is an elliptic surface with a map $f : X \rightarrow \mathbf{P}^1$ onto the projective line. We consider families $g : \mathcal{X} \rightarrow T$ of elliptic surface containing X with a map $f_T : \mathcal{X} \rightarrow \mathbf{P}^1$ which extends f and induces an elliptic fibration on each surface. Subfamilies \mathcal{X}' are to be chosen as differentiable fibre bundles with structure group $Diff_f(X)$, the group of isotopy classes of diffeomorphism which commute with the fibration map up to a diffeomorphism of the base.

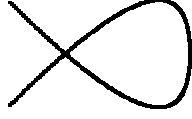
In the given setting there is a natural representation of $Diff_f(X)$ taking values in the mapping class group of the base \mathbf{P}^1 punctured at the singular values, [Bi],

$$M_n^0 = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2, \\ \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 = 1 = (\sigma_1 \cdots \sigma_{n-1})^n \end{array} \right. \right\rangle.$$

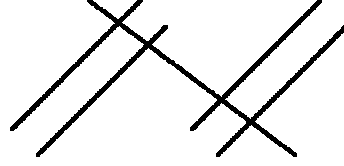
Since there is a natural surjective homomorphism $\pi : Br_n \rightarrow M_n^0$ from the braid group on n strands, we call the associated homomorphism the *braid monodromy* of the family \mathcal{X}' .

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Essentially we consider generic fibrations with singular fibres of type I_1 only. To give some flavour of the general case we allow singular fibres also of type I_0^* , cf. Kodaira's list [Ko], [BPV, p.150]:



I_1



I_0^*

We call a subgroup E of a spherical mapping class group the *braid monodromy group* of a fibration X , if E is the smallest subgroup (w.r.t. inclusion) such that for all admissible \mathcal{X} the image of the braid monodromy is a subgroup of E up to conjugation.

Main Theorem *The braid monodromy group of a regular elliptic fibration X with no singular fibres except $6k$ fibres of type I_1 and l fibres of type I_0^* is a subgroup of M_{6k+l}^0 representing the conjugacy class of*

$$\overline{E}_{6k,l} := \left\langle \sigma_{ij}^{m_{ij}}, i < j \mid m_{ij} = \begin{cases} 1 & \text{if } i, j \leq l \vee i \equiv j(2), i, j > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i, j > l, i \not\equiv j(2) \end{cases} \right\rangle.$$

(Here $\sigma_{i,i+1} := \sigma_i$, while for $j > i + 1$ we define $\sigma_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$.)

Of course it suffices to take an admissible family \mathcal{X} which is topologically versal in the sense, that each admissible family is topologically equivalent to a family induced from \mathcal{X} .

Our interest in this kind of results stems from three sources:

Each family of fibrations we consider is a subfamily of a family of smooth elliptic surfaces. Hence the natural surjection for the fundamental groups of the base spaces induces a map from $\pi_1^{surf.}$ of the base of elliptic surfaces to a quotient of \overline{E} :

$$\begin{array}{ccccc} K & \hookrightarrow & \pi_1^{fib.} & \twoheadrightarrow & \pi_1^{surf.} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{K} & \hookrightarrow & \overline{E} & \twoheadrightarrow & \overline{E}/\overline{K} \end{array}$$

This diagram should be of some help for the understanding of the fundamental group of some suitably defined moduli spaces of regular elliptic surfaces.

There is some analogy to the proposal of Donaldson [Do] to investigate $\pi_1^{surf.}$ by a suitable monodromy map to the symplectic isotopy classes of symplectomorphisms.

In fact $\overline{E}/\overline{K}$ is a quotient of the fundamental group of a discriminant complement for the semi-universal unfolding of a hypersurface singularity $y^3 + x^{6k}$, cf. [Lö]. This observation and the results of Seidel, [Sei], and Auroux, Munoz, Presas, [AMP] point at a striking resemblance of the two homomorphism which deserves further investigations.

Finally we would like to know about the implications of our result in the category of differentiable bundles of elliptic fibrations:

Along with the proof of the theorem we will notice that each mapping class in the braid monodromy group $\overline{E}(X)$ is represented by a diffeomorphism which can be lifted to a diffeomorphism of X inducing the trivial mapping class on some generic fibre. Hence we ask for the converse:

Does every diffeomorphism of X , isotopic to the identity mapping on some generic fibre, induce a mapping class of the punctured base which is in the monodromy group of X ?

A positive answer would yield a topological characterisation of the braid monodromy group!

In fact we show that the group of mapping classes induced in the said way is the image under π of the stabiliser group $Stab_\psi$ of the algebraic monodromy ψ with respect to an appropriate Hurwitz action.

Then we use [Lö] to give an affirmative answer to the question above in case the number of fibres of type I_1 does not exceed 6.

Theorem 1 *Let $\overline{E} := \overline{E}_{6,\ell}$ be the braid monodromy group of a regular elliptic fibration X with no singular fibres except 6 fibres of type I_1 and ℓ fibres of type I_0^* . Then*

$$\overline{E} = \pi(Stab_\psi).$$

bifurcation braid monodromy

With each locally trivial bundle one can associate the structure homomorphism defined on the fundamental group of the base with respect to any base point. It takes values in the mapping classes of the fibre over the base point.

Given a curve C in the affine plane we can take a projection to the affine line which restricts to a finite covering $C \rightarrow \mathbf{C}$. The complement of the curve and its vertical tangents is the total space of a punctured disc bundle over the complement of the branch points in the affine line.

The structure homomorphism of this bundle is called the braid monodromy of the plane curve with respect to the projection, and it can be naturally regarded as a homomorphism from the fundamental group of the branch point complement to the braid group, since the latter is naturally isomorphic to the mapping class group of the punctured disc.

This definition is readily generalized to the case of a divisor in the Cartesian product of the affine or projective line with an irreducible base T . Then the structure homomorphism takes values in a braid group, resp. in a mapping class group of a punctured sphere M_n^0 .

In situations as we are interested in, such a divisor is defined as the locus of critical values of a family of algebraic functions of constant bifurcation degree with values in $\mathbf{L} \cong \mathbf{P}^1$ or \mathbf{C} . Thus we give the relevant definitions:

Definition: A flat family $\mathcal{X} \rightarrow T$ with an algebraic morphism $f : \mathcal{X} \rightarrow \mathbf{L}$ is called a *framed family of functions* (\mathcal{X}, T, f) .

Definition: The *bifurcation set* of a framed family of functions over T is the smallest Zariski closed subset \mathcal{B} in $T \times \mathbf{L}$ such that the diagonal map $\mathcal{X} \rightarrow T \times \mathbf{L}$ is smooth over the complement of \mathcal{B} .

Definition: The *discriminant set* of a framed family of functions over T is the divisor in T such that the bifurcation set \mathcal{B} is an unbranched cover over its complement by the restriction of the natural projection $T \times \mathbf{L} \rightarrow T$.

Definition: A framed family of functions is called *of constant bifurcation degree* if the bifurcation set is a finite cover of T .

Definition: The *bifurcation braid monodromy* of a framed family of functions with constant bifurcation degree over an irreducible base T is defined to be the braid monodromy of \mathcal{B} in $T \times \mathbf{L}$ over T .

Note that this definition of braid monodromy differs slightly from the definition given in the introduction but that the resulting objects are the same.

families of divisors in Hirzebruch surfaces

Given a Hirzebruch surface \mathbf{F}_k with a unique section C_{-k} of selfintersection $-k$, we consider families of divisors on which the ruling of the Hirzebruch surface defines families of functions with constant bifurcation type.

We can pull back divisors from the base along the ruling to get divisors on \mathbf{F}_k which we call vertical, among others the fibre divisor L .

Consider now the family of divisors on \mathbf{F}_k which consist of a vertical part of degree l and a divisor in the complete linear system of $\mathcal{O}_{\mathbf{F}_k}(4C_{-k} + 3kL)$ called the horizontal part. It is a family parameterized by $T = PH^0(\mathcal{O}_{\mathbf{P}^1}(l)) \times PH^0(\mathcal{O}_{\mathbf{F}_k}(4C_{-k} + 3kL))$ with total space

$$\mathcal{D}_{k,l} = \{(x, t) \in \mathbf{F}_k \times T \mid x \in D_t \subset \mathbf{F}_k\}.$$

Let T' be the Zariski open subset of T which is the base of the family $\mathcal{D}'_{k,l}$ of divisors in $\mathcal{D}_{k,l}$ with reduced horizontal part.

Lemma 1 *The ruling on \mathbf{F}_k defines a morphisms $\mathcal{D}'_{k,l} \rightarrow \mathbf{P}^1$ by which it becomes a framed family of functions of constant bifurcation degree.*

Proof: The critical value set of the vertical part of a divisor is the divisor of which it is the pull back, thus it is constant of degree l .

The assumption on reducedness forces the horizontal part to be without fibre components. We may even conclude that a reduced horizontal part consists of C_{-k} and a disjoint divisor which is a branched cover of the base of degree 3. The critical values set is therefore the branch set which is of constant degree $6k$, and we are done. \square

The abstract braid group given by the presentation

$$\text{Br}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \rangle.$$

and the abstract mapping class group M_n^0 given by the presentation in the introduction are naturally identified with the mapping class group of the punctured disc, resp. sphere. Such an identification is given if each σ_i is realised by the half-twist on an embedded arc a_i connecting two punctures provided that $a_i \cap a_{i+1}$ is a single puncture and $a_i \cap a_j$ is empty if $|i - j| \geq 2$.

Proposition 2 *The image of the bifurcation braid monodromy homomorphism of the family $\mathcal{D}'_{k,l}$ is conjugated to the subgroup of M_{6k+l}^0 :*

$$\overline{E}_{6k,l} := \left\langle \sigma_{ij}^{m_{ij}}, i < j \mid m_{ij} = \begin{cases} 1 & \text{if } i, j \leq l \vee i \equiv j(2), i, j > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i, j > l, i \not\equiv j(2) \end{cases} \right\rangle.$$

The proof of this proposition and a couple of preparatory results will take the rest of the section.

First note that our whole concern lies in the understanding of the bifurcation set \mathcal{B} in $T' \times \mathbf{P}^1$ with its projection to T' . As an approximation we will consider families of affine plane curves given by families of polynomials in affine coordinates x, y with the regular map induced by the affine projection $(x, y) \mapsto x$.

Their bifurcation sets are contained in the Cartesian product of the family bases with the affine line \mathbf{C} , and it will soon be shown that this pair can be induced from $(T' \times \mathbf{P}^1, \mathcal{B})$. Eventually we can extract all necessary information from such families to prove our claim.

Lemma 2 *Consider $y^3 - 3p(x)y + 2q(x)$ as a family of polynomial functions $\mathbf{C}^2 \times T \rightarrow \mathbf{C}$ parametrised by a base T of pairs p, q of univariate polynomials. Then the bifurcation set is the zero set of $g(x) := p^3(x) - q^2(x)$, the discriminant set is the zero set of the discriminant of g with respect to x .*

Proof: The bifurcation divisor is cut out by the discriminant polynomial of $y^3 - 3p(x)y + 2q(x)$ with respect to y . The first claim is then immediate since g is proportional to the corresponding Sylvester determinant:

$$\begin{vmatrix} 1 & 0 & -3p & 2q & 0 \\ 0 & 1 & 0 & -3p & 2q \\ 3 & 0 & -3p & 0 & 0 \\ 0 & 3 & 0 & -3p & 0 \\ 0 & 0 & 3 & 0 & -3p \end{vmatrix}$$

For the second claim we note that a pair p, q belongs to the discriminant set if and only if $p^3 - q^2$ has a multiple root hence this locus is cut out by the discriminant of g with respect to x . \square

Lemma 3 *The discriminant locus of a family $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$ is the union of the degeneration component of triples p, q, r defining singular curves and the cuspidal component of triples defining polynomial maps with a degenerate critical point.*

Proof: In general a branched cover of \mathbf{C} has not the maximal number of branch points only if the cover is singular, or the number of preimages of a branch point differs by more than one from the degree of the branching. The second alternative occurs only if there is a degenerate critical point in the preimage or if there are two critical points. Since the last case can not occur in a cover of degree only three we are done. \square

Lemma 4 Given the family $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$ the cuspidal component of the discriminant is the zero set of the resultant of $p(x) + r^2(x)$ and $2q(x) - r^3(x)$ with respect to x .

Its equation - considered a polynomial in the variable λ_0 - is of degree n with coprime coefficients if

$$p(x) = \sum_{i=0}^d \lambda_i x^i, \quad q(x) = x^n + \sum_{i=0}^{n-1} \xi_i x^i, \quad r(x) = \sum_{i=0}^{\lfloor n/3 \rfloor} \zeta_i x^i.$$

Proof: The cuspidal discriminant is the locus of all parameters for which there is a common zero of $f, \partial_y f, \partial_y^2 f$. Since $\partial_y^2 f = 0$ is linear in y , we can eliminate y and get the resultant of $p(x) + r^2(x)$ and $2q(x) - r^3(x)$ with respect to x .

By the degree bound on q and r the discriminant equation is the resultant of a matrix in which the variable λ_0 occurs exactly n times. Moreover the diagonal determines the coefficient of λ_0^n to be a power of $(1 - \zeta_{n/3}^3)$ resp. 1 depending on whether $n/3 \in \mathbf{Z}$ or not. Even in the first case the coefficients are coprime since the resultant is not divisible by $(1 - \zeta_{n/3}^3)$. \square

Lemma 5 For the family $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$ the degeneration component of the discriminant is the locus of triples for which there is a common zero in x, y of the polynomial and its two partial derivatives. Its equation - considered a polynomial in the variable ξ_0 - is monic of degree $2n - 2$ if

$$p(x) = \sum_{i=0}^d \lambda_i x^i, \quad q(x) = x^n + \sum_{i=0}^{n-1} \xi_i x^i, \quad r(x) = \sum_{i=0}^{\lfloor n/3 \rfloor} \zeta_i x^i.$$

Proof: The degeneration locus is given by the Jacobian criterion as claimed. The equation of the discriminant of the subdiagonal unfolding of the quasi-homogeneous singularity $y^3 - x^n$ is known to the quasi-homogeneous and of degree $2n - 2$ in ξ_0 . Since the unfolding over the ξ_0 -parameter is a Morsification the coefficient of ξ_0^{2n-2} must be constant. \square

Lemma 6 The bifurcation braid monodromy of the family $y^3 - 3\lambda y + 2(x^n + \xi)$ maps onto a subgroup of Br_{2n} which is conjugated to the subgroup generated by

$$(\sigma_1 \cdots \sigma_{2n-1})^3, (\sigma_{2n-2, 2n} \cdots \sigma_{2, 4})^{n+1}, (\sigma_{2n-3, 2n-1} \cdots \sigma_{1, 3})^{n+1}.$$

Proof: As one can show with the help of the preceding lemmas, the discriminant locus is the union of the degeneration locus and the cuspidal component which are cut out respectively by the polynomials $\lambda^3 - \xi^2$ and λ .

By Zariski/van Kampen the fundamental group of the complement with base point $(\lambda, \xi) = (1, 0)$ is generated by the fundamental group of the complement restricted to the line $\lambda = 1$ and the homotopy class of a loop which links the line $\lambda = 0$ once.

For the pair $(1, 0)$ the set of regular values of the polynomial consists of the affine line punctured at the $(2n)$ -th roots of unity, which we number counterclockwise, 1 the first puncture. To express the bifurcation braid group in terms of abstract generators, we identify the elements σ_i with the half twist on the circle segment between the i -th and $i + 1$ -st puncture.

For the line $\lambda = 1$ the bifurcation locus is given by $(x^n + \xi - 1)(x^n + \xi + 1)$. This locus is smooth but branches of degree n over the base at $\xi = \pm 1$. The corresponding monodromy transformations are the second and third transformation given in the claim.

Associated to the degeneration path $(\lambda, \xi) = (1 - t, \sqrt{-1}t)$, $t \in [0, 1]$ there is a loop in the complex line $\lambda = 1 + \sqrt{-1}\xi$ which links the line $\lambda = 0$. For this degeneration the bifurcation divisor is regular and contains points of common absolute value determined by t only, except for $t = 1$ where it has n ordinary cusps with horizontal tangent cone. Since a cusp corresponds to a triple half twist and the first and second puncture merge in the degeneration, the monodromy transformation for our loop is the first braid of the claim. \square

Lemma 7 *The bifurcation braid monodromy of the family $y^3 - 3\lambda y + 2(x^n + \xi + \varepsilon x)$, ε small and fix, is in the conjugation class of the subgroup of the braid group generated by*

$$(\sigma_1 \sigma_3 \cdots \sigma_{2n-1})^3, \sigma_{i,i+2}, i = 1, \dots, 2n - 2.$$

Proof: The discriminant locus in the λ, ξ parameter plane consists again of the cuspidal component $\lambda = 0$ and the degeneration component.

Since the perturbation ε is arbitrarily small, some features of the family of lemma 6 are preserved. The conclusion of the Zariski/van Kampen argument still holds, each braid group generators σ_i is now realized as half twist on segments of a slightly distorted circle, and the loop linking $\lambda = 0$ is only slightly perturbed. So the monodromy transformation associated to this loop is formally the same as before, the first braid in the claim.

The dramatic change occurs in the bifurcation curve over the line $\lambda = 1$. Now the bifurcation locus is the union of two disjoint smooth components each of which branches simply of degree n with all branch points near $\xi = 1$, resp. $\xi = -1$. Since the local model $x^n + \varepsilon x$ has the full braid group as its monodromy group, the monodromy along $\lambda = 1$ is generated by the elements $\sigma_{i,i+2}$ as claimed. \square

Lemma 8 *The bifurcation braid monodromy of the family*

$$y^3 - 3(\lambda + \lambda_1 x)y + 2(x^n + \xi + \xi_1 x)$$

is in the conjugation class of the subgroup generated by

$$\sigma_i^3, i \equiv 1(2), \sigma_{i,i+2}, 0 < i < 2n - 1.$$

Proof: The components of the discriminant are the degeneration component and the cuspidal component.

The line $\lambda = 1, \lambda_1 = 0, \xi_1 = \varepsilon$ small and fix, is generic for the degeneration component and we may conclude from lemma 7 that there are elements in the fundamental group of the discriminant complement with respect to $(\lambda, \lambda_1, \xi, \xi_1) = (1, 0, 0, 0)$ which map to $\sigma_{i,i+2}$ as in lemma 7.

Since the line $\xi = i, \xi_1 = \lambda_1 = 0$ is transversal for the cuspidal component so are parallel lines with $\lambda_1 = \varepsilon'$ small and fix. The bifurcation curve is then given by $(\lambda + \varepsilon' x)^3 = (x^n + \sqrt{-1})^2$. For $\lambda = 0$ the critical values are distributed in pairs along a circle in the affine line which merge pairwise for $\lambda_1 \rightarrow 0$.

But for ε' sufficiently small and by varying λ to an appropriate small extend such that the degeneration component is not met, there are n obvious degenerations when λ is of the same modulus as ε' and $\lambda + \varepsilon'x$ is a factor of $x^n + \sqrt{-1}$. By the local nature of the degeneration a degree argument shows that these are all possibilities. Moreover one can easily see that the corresponding monodromy transformations are triple half twist for each one of the pairs.

Moreover these twists are transformed to the transformations $\sigma_{i,i+1}^3$, i odd, when transported along $(1-t, \sqrt{-1}t, 0, \varepsilon't)$, $t \in [0, 1]$ to the chosen reference point.

The monodromy group is thus completely determined since the fundamental group is generated by elements which map to the given group under the monodromy homomorphism. \square

Lemma 9 *Let a family of plane polynomials be given which is of the form*

$$y^3 + 3\left(\sum_{i=0}^{d_r} \zeta_i x^i\right)y^2 - 3\left(\sum_{i=0}^{d_p} \lambda_i x^i\right)y + 2\left(x^n + \sum_{i=0}^{n-1} \xi_i x^i\right),$$

$$2 \leq n, 0 < 3d_p \leq 2n, 3d_r \leq n.$$

Then the bifurcation braid monodromy group is in the conjugation class of the subgroup of the braid group generated by

$$\sigma_i^3, i \equiv 1(2), \sigma_{i,i+2}, 0 < i < 2n-1.$$

Proof: Since the family considered in the previous lemma is a subfamily now and has the claimed monodromy, we have to show that the new family has no additional monodromy transformations.

In the proof above we have seen that the cuspidal component is cut in n points by a line in λ_0 direction. The component is reduced since its multiplicity at the origin is n , too, by lemma 4. The degeneration component is reduced by the analogous argument relying on lemma 5 and transversally cut in $2n-2$ points by a line in ξ_0 direction. Hence by Zariski/van Kampen arguments as proved in [Be] the fundamental group of the discriminant complement of the subfamily surjects onto the fundamental group of the family considered now. \square

Lemma 10 *Define a subgroup of the braid group Br_{2n+l} :*

$$E_{2n,l} := \left\langle \sigma_{i,j}^{m_{ij}}, i < j \mid m_{ij} = \begin{cases} 1 & \text{if } i, j \leq l \vee i \equiv j(2), i > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i, j > l \wedge i \not\equiv j(2) \end{cases} \right\rangle.$$

Then the same subgroup is generated also by

$$\sigma_i, i < l, \sigma_{i,i+2}, l < i, \sigma_{i,j}^2, i \leq l < j, \sigma_i^3, i > l, i \not\equiv l(2).$$

Proof: We have to show that the redundant elements can be expressed in the elements of the bottom line. This is immediate from the following relations ($i < j$):

$$\begin{aligned} \sigma_{i,j} &= \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}, & j \leq l, \\ \sigma_{i,j} &= \sigma_{j-2,j} \cdots \sigma_{i+2,i+4} \sigma_{i,i+2} \sigma_{i+2,i+4} \cdots \sigma_{j-2,j}, & l < i, i \equiv j(2), \\ \sigma_{i,j}^3 &= \sigma_{j-2,2} \cdots \sigma_{i+1,i+3} \sigma_i^3 \sigma_{i+1,i+3} \cdots \sigma_{j-2,j}, & l < i, i \not\equiv j(2), \\ \sigma_{i,j}^3 &= \sigma_{j-2,2} \cdots \sigma_{i+1,i+3} \sigma_{i-1,i+1} \sigma_{i-1,i+1}^3 \sigma_{i-1,i+1} \sigma_{i+1,i+3} \cdots \sigma_{j-2,j}, & l < i, j \not\equiv l(2). \end{aligned}$$

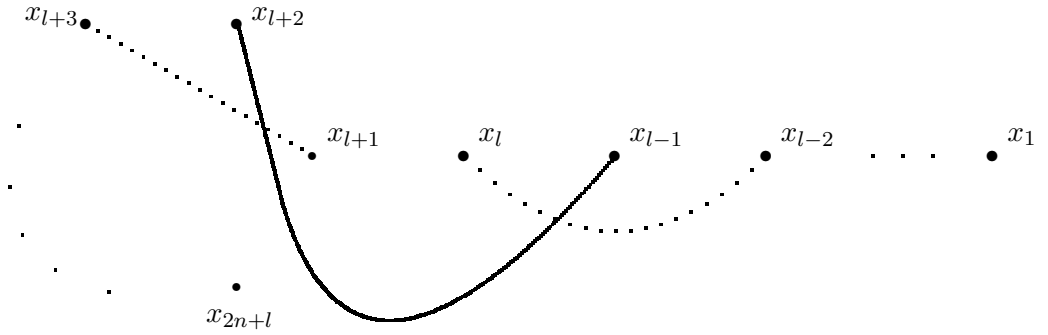
\square

Lemma 11 Consider the family $(y^3 - 3p(x)y + 2q(x))a(x)$ parametrised by triples p, q, a , with p from the vector space of univariate polynomials of degree at most $2n/3$, q, a from the affine space of monic polynomials of degree n and l respectively. Then the subgroup $E_{2n,l}$ of Br_{2n+l} is conjugate to a subgroup of the image of the bifurcation braid monodromy.

Proof: We choose our reference divisor to be $(y^3 - 3y + 2x^n) \prod_i^l (x - l - 2 + i)$ with corresponding bifurcation set $x_i = l + 2 - i, i \leq l$ on the real axis and $x_{l+1} = 1$ and the $x_i, i > l + 1$ equal to the $2k^{\text{th}}$ -roots of unity in counterclockwise numbering. We identify the elements $\sigma_{i,j}$ of the braid group with the half twist on arcs between x_i, x_j , which are chosen to be

- i) a circle segment through the lower half plane, if $i, j \leq l$,
- ii) a circle secant in the unit disc, if $i, j > l$,
- iii) the union of a secant in the unit disc to a point on its boundary between x_{2n+l} and 1 with an arc through the lower half plane, if $i \leq l < j$.

(Of each kind we have depicted one in the following figure.)



Since keeping the horizontal part $y^3 - 3y + 2x^n$ fix, the bifurcation divisor of the vertical is equivalent to that of the universal unfolding of the function x^l we have the elements $\sigma_i, i < l$ in the braid monodromy. These elements are obtained for example in families

$$a(x) = ((x - l + i - 3/2)^2 + \lambda) \prod_{j \neq i, i+1}^l (x - l - 2 + j).$$

The second set of elements, $\sigma_{i,j}^2, i \leq l < j$ is obtained by families of the kind

$$(y^3 - 3y + 2x^n)(x - l - 2 + i - \lambda) \prod_{j \neq i}^l (x - l - 2 + j)$$

since the zero $l + 2 - i + \lambda$ may trace any given path in the range of the projection, in particular that around an arc on which the full twist $\sigma_{i,j}^2$ is performed.

Finally varying the horizontal part as in lemma 8 while keeping the $a(x)$ factor fix proves that the braid group elements $\sigma_{i,i+2}, l < i$ and $\sigma_i, l < i, i \neq l(2)$ are in the image of the monodromy. So we may conclude that this image contains $E_{2n,l}$ up to conjugacy. \square

Proof of prop. 2: Denote by S the Zariski open subset of T' which parameterizes divisors of the family $\mathcal{D}'_{k,l}$ which have no singular value at a point $\infty \in \mathbf{P}^1$. The corresponding family in $\mathbf{F}_k \times S$ may then be restricted to a family $\mathcal{F}_{k,l}$ in $\mathbf{C} \times \mathbf{C} \times S$, where \mathbf{F}_k is trivialized as $\mathbf{C} \times \mathbf{C}$ in the complement of the negative section C_{-k} and the fibre over ∞ . By construction $\mathcal{F}_{k,l}$ has constant bifurcation degree. Consider now the family of polynomials

$$(y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)) a(x),$$

where r, p, q, a are taken from the family of all quadruples of polynomials in one variable subject to the conditions that

- i) r, p are of respective degrees k and $2k$,
- ii) q is monic of degree $3k$, a is monic of degree l
- iii) the discriminant of $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$ is not identically zero.

This family can be naturally identified with $\mathcal{F}_{k,l}$. By lemma 11, up to conjugacy, $E_{6k,l}$ is contained in the monodromy image $\rho(\pi_1(S \setminus \text{Discr}(\mathcal{F}_{k,l})))$.

For the converse we note that the bifurcation set of the family decomposes into the bifurcation sets Bif_h of the horizontal part $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$ and Bif_v of the vertical part $a(x)$. Hence the monodromy is contained in the subgroup $\text{Br}_{(6k,l)}$ of braids which do not permute points belonging to different components. $\text{Br}_{(6k,l)}$ has natural maps to Br_{6k} and Br_l which commute with the braid monodromies of both bifurcation set considered on their own.

The discriminant decomposes into the discriminants of Bif_h , Bif_v and the divisor of parameters for which $\text{Bif}_h \cap \text{Bif}_v$ not empty. They give rise in turn to braids which can be considered as elements in

$$\text{Br}_{6k}, \text{Br}_l \text{ resp. } \text{Br}_{(6k,l)}^{0,0} := \{\beta \in \text{Br}_{(6k,l)} \mid \beta \text{ trivial in } \text{Br}_{6k} \times \text{Br}_l\}.$$

Now with lemma 10 we can identify $E_{6k,l}$ as the subgroup of Br_{6k+l} generated by $E_{6k} \subset \text{Br}_{6k}$, Br_l and $\text{Br}_{(6k,l)}^{0,0}$ which are generated in turn by the elements

$$\{\sigma_{i,i+2}, \sigma_i^3, l < i\}, \{\sigma_i, i < l\}, \{\sigma_{i,j}^2, i \leq l < j\} \text{ resp.}$$

And by lemma 9 the image can not contain more elements.

Since the bifurcation diagram of $\mathcal{F}_{k,l}$ embeds in the bifurcation diagram of $\mathcal{D}'_{k,l}$ with complement of codimension one, there is a commutative diagram

$$\begin{array}{ccc} \pi_1(S \setminus \text{Discr}(\mathcal{F}_{k,l})) & \longrightarrow & \pi_1(T' \setminus \text{Discr}(\mathcal{D}'_{k,l})) \\ \downarrow & & \downarrow \\ E_{6k,l} & & \\ \downarrow & & \\ \text{Br}_{6k+l} & \longrightarrow & M_{6k+l}^0 \end{array}$$

from which we read off our claim. □

Corollary 1 *For any element β in the braid monodromy group of $\mathcal{D}'_{k,l}$ there is a diffeomorphism of the base \mathbf{P}^1 which fixes a neighbourhood of $\infty \in \mathbf{P}^1$ and which represents the mapping class β .*

Proof: The element β is image of an element β' in the braid monodromy of the bifurcation diagram of $\mathcal{F}_{k,l}$. The bifurcation set does not meet the boundary so integration along a suitable vector field yields a realisation of β' as a diffeomorphism acting trivially on a neighbourhood of the boundary. Its trivial extension to the point ∞ is the diffeomorphism sought for. \square

families of elliptic surfaces

In this section we start investigating families of regular elliptic surfaces for which the type of singular fibres is restricted to I_1 and I_0^* . We will go back and forth between a family of elliptic fibrations, its associated family of fibrations with a section and a corresponding Weierstrass model of the latter, so we note some of their properties:

Proposition 3 *Given a family of elliptic fibrations with constant bifurcation type over an irreducible base T , there is a family of elliptic fibrations with a section, such that the bifurcation sets of both families coincide.*

Proof: Given a family as claimed there is the associated family of Jacobian fibrations, cf. [FM, I.5.30]. The bifurcation sets of both families coincide. \square

In turn, for each family of elliptic fibrations with a section there is a corresponding family of Weierstrass fibrations, cf. Miranda [M].

A regular Weierstrass fibration W is defined by an equation

$$wz^2 = 4y^3 - 3Pw^2y + 2Qw^3$$

in the projectivisation of the vector bundle $\mathcal{O} \oplus \mathcal{O}(2\chi) \oplus \mathcal{O}(3\chi)$ over the projective line \mathbf{P}^1 where χ is the holomorphic Euler number of the fibration, w, y, z are 'homogeneous coordinates' of the bundle, and P, Q are sections of $\mathcal{O}(4\chi), \mathcal{O}(6\chi)$ respectively.

So W is a double cover of the Hirzebruch surface $\mathbf{F}_{2\chi} = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2\chi))$ branched along the section $\sigma_{2\chi}$ and the divisor in its complement $\mathcal{O}(2\chi)$ defined by the equation $y^3 - 3Py + 2Q = 0$.

A framed family of Weierstrass fibrations over a parameter space T is a given by data as before where now P, Q are sections of the pull backs to $T \times \mathbf{P}^1$ of $\mathcal{O}(4\chi), \mathcal{O}(6\chi)$ such that for each parameter $\lambda \in T$ they define a Weierstrass fibration. In the sequel P, Q are referred to as the coefficient data of the Weierstrass family.

Lemma 12 *Let \mathcal{W} be the Weierstrass family associated to a framed family over T of regular elliptic fibrations in which all surfaces have no singular fibres except for l of type I_0^* and $6k$ of I_1 with coefficient data P, Q , then there are three families of sections a, p, q of $\mathcal{O}(l), \mathcal{O}(2k), \mathcal{O}(3k)$ respectively, such that p, q have no common zero,*

$$p \cdot a^2 = P, \quad q \cdot a^3 = Q,$$

and the bifurcation set is given by

$$a(p^3 - q^2) = 0 \subset T \times \mathbf{P}^1.$$

Proof: By the classification of Kas [Ka] at base points of regular fibres the discriminant $P^3 - Q^2$ does not vanish, at base points of fibres of type I_1 the discriminant vanishes but neither P nor Q and at base points of fibres of type I_0^* the vanishing order of P is two, the vanishing order of Q is three.

Since by hypothesis the locus of base points of singular fibres of type I_0^* form a family of point divisors of degree l there is a section a of $\mathcal{O}(l)$ such that P has a factor a^2 and Q a factor a^3 .

With $\deg P = 2(l + k)$, $\deg Q = 3(l + k)$ we get the other degree claims.

Finally the discriminant of the Weierstrass fibration is given by $P^3 - Q^2$ which has – by the above – the same zero set as $a(p^3 - q^2)$. \square

Remark: In the situation of the lemma, a family of divisors is given for \mathbf{F}_k by the equation $a(y^3 - 3pw^2y + 2qw^3) = 0$, a cutting out the vertical part. The double cover along this divisor is a family of fibrations obtained from the original family by contracting all smooth rational curves of selfintersection -2 , of which there are four for each fibre of type I_0^* .

We are now prepared to come back to the main theorem:

Proof of the main theorem: Given any framed family of regular elliptic fibrations containing X we consider a Weierstrass model \mathcal{W} of the associated Jacobian family. Since \mathcal{W} is again framed there is an induced family of divisors on a Hirzebruch surface obtained as before.

This family of divisors is a pull back from the space $\mathcal{D}_{k,l}$ so the monodromy is a subgroup of the bifurcation monodromy of Hirzebruch divisors.

On the other hand for the family of triples of polynomials $p(x), q(x), a(x)$ with p of degree at most $2k$ and q, a monic of degree $3k$ respectively l , we can form the family given by

$$z^2 = y^3 - 3p(x)a^2(x) + 2q(x)a^3(x),$$

which is Weierstrass in the complement of parameters where $a(x)(p^3(x) - q^2(x))$ has a multiple root or vanishes identically. At least after suitable base change, cf. [FM, p. 163], this Weierstrass family has a simultaneous resolution yielding a family $\mathcal{X}_{k,l}$ of elliptic surfaces with a section.

The Jacobian of X is contained in $\mathcal{X}_{k,l}$, since its Weierstrass data consist of sections P, Q which are factorisable as a^2p, a^3q according to lemma 12 and after the choice of a suitable ∞ this data can be identified with polynomials in this family.

The fibration X is deformation equivalent to its Jacobian with constant local analytic type, cf. [FM, thm. I.5.13] and hence of constant fibre type. The monodromy group therefore contains the bifurcation monodromy group of divisors on Hirzebruch surfaces $\mathcal{D}_{k,l}$ and so the two groups even coincide. \square

Regarding elements in the braid monodromy as mapping classes again they can be shown to be induced by diffeomorphism of the elliptic fibration, but more is true in fact:

Proposition 4 *For each braid β in the framed braid monodromy group there is a diffeomorphism of the elliptic fibration which preserves the fibration, induces β on the base and the trivial mapping class on some fibre.*

Proof: As we have seen in the corollary to prop. 2 we can find a representative φ for the braid β by careful integration of a suitable vector field such that φ is the identity next to a point ∞ .

In [FM, II.1.2] there is a proof for families of nodal elliptic fibrations and sufficient hints for more general families of constant singular fibre types, that a horizontal vector field on the total family can be found which fails to be a lift only in arbitrarily small neighbourhoods of singular points on singular fibres. Integration of such a vector field yields a diffeomorphism $\tilde{\varphi}$ which is a lift of φ .

We have seen that the monodromy generators arising from the horizontal part can be realized over a suitable polydisc parameter space, cf. lemma 8. Since the vertical part as in lemma 11 does not have any effect on the fibre F_∞ over ∞ we can conclude that this fibration family is the trivial family next to F_∞ . So we apply the argument above to get a lift $\tilde{\varphi}$ which induces the trivial mapping class on F_∞ . \square

Hurwitz stabilizer groups

In this section we determine the stabilizers of the action of the braid group Br_n on homomorphisms defined on the free group F_n generated by elements t_1, \dots, t_n . The action is given by precomposition with the Hurwitz automorphism of F_n associated to a braid in Br_n :

$$\text{Br}_n \rightarrow \text{Aut } F_n : \quad \sigma_{i,i+1} \mapsto \left(t_j \mapsto \begin{cases} t_j & j \neq i, i+1 \\ t_i t_j t_i^{-1} & j = i \\ t_i & j = i+1 \end{cases} \right).$$

We start with a result from [Lö]:

Proposition 5 *Let $F_n := \langle t_i, 1 \leq i \leq n \mid \rangle$ be the free group on n generators, define a homomorphism $\phi_n : F_n \rightarrow \text{Br}_3 = \langle a, b \mid aba = bab \rangle$ by*

$$\phi_n(t_i) = \begin{cases} a & i \text{ odd} \\ b & i \text{ even} \end{cases}$$

and let Br_n act on homomorphisms $F_n \rightarrow \text{Br}_3$ by Hurwitz automorphisms of F_n . Then the stabilizer group Stab_{ϕ_n} contains the braid subgroup

$$E_n = \langle \sigma_{i,j}^{m_{ij}} \mid m_{ij} = 1, 3 \text{ if } j \equiv i, \text{ resp. } i \not\equiv j \pmod{2} \rangle$$

with $E_n = \text{Stab}_{\phi_n}$, if $n \leq 6$.

Note that the action in [Lö] was defined on tuples $(\phi_n(t_1), \dots, \phi_n(t_n))$ but that it is obviously equivalent to the action considered here.

This result can now be applied to find stabilizers of similar homomorphisms:

Proposition 6 *Let $F_n := \langle t_i, 1 \leq i \leq n \mid \rangle$ be the free group on n generators, define a homomorphism $\psi_n : F_n \rightarrow \text{SL}_2 \mathbf{Z}$ by*

$$\psi_n(t_i) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & i \text{ odd} \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & i \text{ even} \end{cases}$$

and let Br_n act on homomorphisms $F_n \rightarrow \text{SL}_2 \mathbf{Z}$ by Hurwitz automorphisms of F_n . Then the stabilizer group Stab_{ψ_n} of ψ_n is equal to the stabilizer group Stab_{ϕ_n} of ϕ_n .

Proof: Both groups, $\mathrm{SL}_2\mathbf{Z}$ and Br_3 , are central extensions of $\mathrm{PSL}_2\mathbf{Z}$, and both ϕ_n and ψ_n induce the same homomorphism $\chi_n : F_n \rightarrow \mathrm{PSL}_2\mathbf{Z}$. Of course Stab_χ contains Stab_ϕ and Stab_ψ and thus our claim is proved as soon as we can show the opposite inclusions.

First note that the braid action defined on homomorphisms as above is equivalent to the Hurwitz action on the tuples of images of the specified generators $t_i \in F_n$, hence the braid action will not change the conjugation class of these images.

Now let β be a braid in Stab_χ . Then $\phi \circ \beta(t_i) = (ab)^{3k_i} \phi(t_i)$ since $(ab)^3$ is the fundamental element of Br_3 which generates the center of Br_3 and thus the kernel of the extension $\mathrm{Br}_3 \rightarrow \mathrm{PSL}_2\mathbf{Z}$. The degree homomorphism $d : \mathrm{Br}_3 \rightarrow \mathbf{Z}$ is a class function with value one on all $\phi(t_i)$, hence $d((ab)^{3k_i}) = 0$. Since $d(ab) = 2$ we conclude $k_i = 0$ and $\beta \in \mathrm{Stab}_\phi$.

Similarly we have $\psi \circ \beta(t_i) = \pm \psi(t_j)$ for $\beta \in \mathrm{Stab}_\chi$. Since the trace is a class function on $\mathrm{SL}_2\mathbf{Z}$ which has value 2 on all $\psi(t_i)$ while it is -2 on $-\psi(t_i)$, we also get $\beta \in \mathrm{Stab}_\psi$. \square

Proposition 7 *Let $F_n := \langle t_i, 1 \leq i \leq n \mid \rangle$ be the free group on $n = l + l'$ generators, define a homomorphism $\psi_{l,l'} : F_n \rightarrow \mathrm{SL}_2\mathbf{Z}$ by*

$$\psi_{l,l'}(t_i) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & i > l, i \not\equiv l \pmod{2} \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & i > l, i \equiv l \pmod{2} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & i \leq l \end{cases}$$

and let Br_n act on homomorphisms $F_n \rightarrow \mathrm{SL}_2\mathbf{Z}$ by Hurwitz automorphisms of F_n . Then the stabilizer group $\mathrm{Stab}_{\psi_{l,l'}}$ of $\psi_{l,l'}$ is generated by the image of $\mathrm{Stab}_{\psi_{l'}}$ under the inclusion $\mathrm{Br}_{l'} \hookrightarrow \mathrm{Br}_n$ mapping to braids with only the last l' strands braided and

$$E_{l,l'} := \left\langle \sigma_{ij}^{m_{ij}}, 1 \leq i < j \leq n \mid m_{ij} = \begin{cases} 1 & \text{if } j \leq l \vee i \equiv j(2), i > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i > l, i \not\equiv j(2) \end{cases} \right\rangle.$$

If $l' \leq 6$ then even $\mathrm{Stab}_{\psi_{l,l'}} = E_{l,l'}$.

Proof: Again we argue with the equivalent Hurwitz action on images of the generators. First we consider the induced action on conjugacy classes. On n -tuples of conjugacy classes the Hurwitz action induces an action of Br_n through the natural homomorphism π to the permutation group S_n . Since the tuple induced from ψ consists of l copies of the conjugacy class of $-id$ followed by l' copies of the distinct conjugacy class of $\psi(t_1)$, the associated stabilizer group is $\tilde{E} := \pi^{-1}(S_l \times S_{l'})$, and as in [KL] one can check that

$$\tilde{E} = \langle \sigma_{ij}, i < j \leq l \text{ or } l < i < j; \tau_{ij} := \sigma_{ij}^2, i \leq l < j \rangle.$$

So as a first step we have Stab_ψ contained in \tilde{E} .

Since $-id$ is central it is the only element in its conjugacy class and we may conclude that the \tilde{E} orbit of ψ contains only homomorphisms which map the first l generators onto $-id$. With a short calculation using that $-id$ is a central involution

we can check that the τ_{ij} act trivially on such elements:

$$\begin{aligned}
& \tau_{ij}(-id, \dots, -id, M_{l+1}, \dots, M_n) \\
&= \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^2 \sigma_{j-1} \cdots \sigma_{i+1}(-id, \dots, -id, M_{l+1}, \dots, M_n) \\
&= \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^2(-id, \dots, -id, M_{l+1}, \dots, M_{j-1}, -id, M_j, \dots, M_n) \\
&= \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}(-id, \dots, -id, M_{l+1}, \dots, M_{j-1}, -id, M_j, \dots, M_n) \\
&= (-id, \dots, -id, M_{l+1}, \dots, M_n)
\end{aligned}$$

Therefore given $\beta \in \tilde{E}$ as a word w in the generators σ_{ij}, τ_{ij} of \tilde{E} the action of β on ψ is the same as that of β' where β' is given by a word w' obtained from w by dropping all letters τ_{ij} . By the commutation relations of the σ_{ij} we may collect all letters $\sigma_{ij}, i, j \leq l$ to the right of letters $\sigma_{ij}, i, j > l$ without changing β' and get a factorization $\beta' = \beta'_1 \beta'_2$ with $\beta'_1 \in \text{Br}_l, \beta'_2 \in \text{Br}_{l'}$.

Hence $\beta \in \tilde{E}$ acts trivially on ψ if and only if $\beta'_1 \beta'_2$ does so if and only if β'_2 acts trivially on $\psi_{l'}$. Thus $\text{Stab}_{\psi_{l'}}$ is generated by the τ_{ij} the $\sigma_{ij}, i, j \leq l$ and the $\beta'_2 \in \text{Stab}_{\psi_{l'}}$. Both conclusions of the proposition then follow since $\sigma_{ij}, i, j > l$ are contained in $\text{Stab}_{\psi_{l'}}$ and since they are even generators if $l' \leq 6$, prop. 5. \square

mapping class groups of elliptic fibrations

We return to elliptic fibrations and obtain some results concerning mapping classes of elliptic fibrations. In fact we need to enrich the structure a bit:

Definition: A *marked elliptic fibration* is an elliptic fibration with a distinguished regular fibre, $f : X, F \rightarrow B, b_0$, which can be thought of as given by a *marking* $F \hookrightarrow E$.

Definition: A *fibration preserving map* of a marked elliptic surface $f : X, F \rightarrow B, b_0$ is a homeomorphism $\tilde{\varphi}_X$ of X such that $f \circ \tilde{\varphi}_X = \varphi_{B, b_0} \circ f$ for a homeomorphism φ_{B, b_0} of (B, b_0) and such that $\tilde{\varphi}_X|_F$ is isotopic to the identity on F . The map φ_{B, b_0} is called the *induced base homeomorphism*.

An induced homeomorphism necessarily preserves the set $\Delta(f)$ of singular values of the fibration map f and therefore can be regarded as a homeomorphism of the punctured base $B, \Delta(f)$ preserving the base point.

On the other hand with each elliptic fibration $f : X \rightarrow B$ we have a torus bundle over the complement B^0 of $\Delta(f)$. Its structure homomorphism is the natural map

$$\psi : \pi_1(B^0, b_0) \longrightarrow \text{Diff}(F)$$

to the group of isotopy classes of diffeomorphisms of the distinguished fibre.

Lemma 13 *Let $X, F \rightarrow B, b_0$ be a marked elliptic fibration and β a braid represented by a homeomorphism of B^0, b_0 . Then there is a fibration preserving map $\tilde{\varphi}_X$ inducing $\pi(\beta)$ if and only if β stabilises the structure map of the associated torus bundle.*

Proof: A fibration preserving homeomorphism $\tilde{\varphi}$ of an unmarked elliptic surface induces a map φ_B of the punctured base B^0 . By the classification of torus bundles there exists then a commutative diagram

$$\begin{array}{ccc} \pi_1(B^0, b_0) & \xrightarrow{(\varphi_B)_*} & \pi_1(B^0, \varphi_B(b_0)) \\ \downarrow \psi_{b_0} & & \downarrow \psi_{\varphi(b_0)} \\ \text{Diff}(F) & \xrightarrow{(\tilde{\varphi}|_F)_*} & \text{Diff}(\tilde{\varphi}(F)) \end{array}$$

But the result of Moishezon [Moi, p. 169] implies that the reverse implication is true in the absence of multiple fibres.

If now $\tilde{\varphi}$ is a fibration preserving homeomorphism of a marked elliptic surface then the bottom map is the identity and the claim is immediate. \square

In order to relate to the results of the last section we use surjective maps

$$F_{6k+l} \rightarrow \pi_1(B, b_0)$$

provided by a choice of geometric basis, i.e. an ordered system of generators which are simultaneously represented by disjoint loops, each going around a single element of $\Delta(f)$.

Lemma 14 *Given a marked elliptic fibration $X, F \rightarrow B, b_0$ with singular fibres only of types I_1, I_0^* and an isomorphism $\text{Diff}(F) \cong \text{SL}_2\mathbf{Z}$, there is a choice of geometric basis for $\pi_1(B^0, b_0)$ such that the structure homomorphism of the associated bundle is $\psi_{6k,l}$.*

Proof: The proof proceeds along the lines of Moishezon's proof [Moi], cf. [FM], for the normal form of an elliptic surface with only fibres of type I_1 . The same strategy leads to our claim since fibres of type I_0^* have local monodromy in the center of $\text{SL}_2\mathbf{Z}$. \square

By now we have finally got all necessary results to prove theorem 1 as stated in the introduction.

Proof of theorem 1: As before M_{6+l}^0 denotes the mapping class group of $B, \Delta(f)$. We have previously shown that the mapping classes induced by fibration preserving maps are represented by braids acting trivially on the structure homomorphism of the torus bundle given with the elliptic fibration, lemma 13.

By lemma 14 and prop. 7 the corresponding group is conjugation equivalent to $\pi(E_{6,l}) = \overline{E}_{6,l}$. On the other hand the monodromy group is in the conjugation class of $\overline{E}_{6,l}$ by the main theorem. Moreover for each mapping class of the monodromy group there is by prop. 4 a fibration preserving diffeomorphism, so we get an inclusion and hence both groups coincide as claimed. \square

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